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# BRIEF COMMUNICATIONS

## A CHARACTERIZATION OF TOTALLY UMBILICAL HYPERSURFACES OF A SPACE FORM BY GEODESIC MAPPING

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The idea of considering the second fundamental form of a hypersurface as the first fundamental form of another hypersurface has found very useful applications in Riemannian and semi-Riemannian geometry, especially when trying to characterize extrinsic hyperspheres and ovaloids. Recently, T. Adachi and S. Maeda gave a characterization of totally umbilical hypersurfaces in a space form by circles. In our paper, we give a characterization of totally umbilical hypersurfaces of a space form by means of geodesic mapping.

### 1. Introduction

Let $M_n$ and $M'_n$ be two hypersurfaces of a space form $\bar{M}_{n+1}$ [3–5] and let $g,\ g'$ and $\bar{g}$ be the respective positive-definite metric tensors. By $\nabla,\ \nabla'$, and $\bar{\nabla}$ we denote the corresponding connections induced by $g,\ g'$, and $\bar{g}$.

In the present paper, we choose the first fundamental form of $M'_n$ as

$$g' = e^{2\sigma}\omega, \tag{1.1}$$

where $\omega$ is the second fundamental form of $M_n$ which is supposed to be positive-definite and $\sigma$ is a differentiable function defined on $M_n$.

Let $\{x^i\},\ \{x'^i\}$, and $\{y^\alpha\}$ be the respective coordinate systems in $M_n,\ M'_n$, and $\bar{M}_{n+1}$ and let $f$ be a one-to-one differentiable mapping of $M_n$ upon $M'_n$ defined by

$$x'^i = f^i(x^1, x^2, \ldots, x^n), \quad i = 1, 2, \ldots, n, \tag{1.2}$$

where $f^i$ are smooth functions defined on $M_n$. Also let the corresponding Jacobian be nonvanishing. Then it is clear that the corresponding points of $M_n$ and $M'_n$ are represented by the same set of coordinates and that the coordinate vectors are in correspondence.

Let $\bar{R},\ R$, and $R'$ be the covariant curvature tensors of $\bar{M}_{n+1},\ M_n$, and $M'_n$, respectively, and let $\bar{K}$ be the Riemannian curvature of $\bar{M}_{n+1}$.

Thus, we have[1]

$$\bar{R}_{\beta\gamma\delta\epsilon} = \bar{K}(\bar{g}_{\beta\delta}\bar{g}_{\gamma\epsilon} - \bar{g}_{\beta\epsilon}\bar{g}_{\gamma\delta}). \tag{1.3}$$

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[1] In what follows, the Latin indices $i, j, k, \ldots$ run from $1$ to $n$, while the Greek indices $\alpha,\ \beta$, and $\gamma$ run from $1$ to $n+1$.

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On the other hand, under the condition (1.3), the Codazzi equations

$$\nabla_k\omega_{ij} - \nabla_j\omega_{ik} + \bar{R}_{\beta\gamma\delta\epsilon}N^\beta \frac{\partial y^\gamma}{\partial x^i}\frac{\partial y^\delta}{\partial x^j}\frac{\partial y^\epsilon}{\partial x^k} = 0$$

and the Gauss equation

$$R_{ijkl} = \bar{R}_{\beta\gamma\delta\epsilon}\frac{\partial y^\beta}{\partial x^i}\frac{\partial y^\gamma}{\partial x^j}\frac{\partial y^\delta}{\partial x^k}\frac{\partial y^\epsilon}{\partial x^l} + (\omega_{ik}\omega_{jl} - \omega_{il}\omega_{jk})$$

transform, respectively, into

$$\nabla_k\omega_{ij} - \nabla_j\omega_{ik} = 0 \tag{1.4}$$

and

$$R_{ijkl} = \bar{K}(g_{ik}g_{jl} - g_{il}g_{jk}) + (\omega_{ik}\omega_{jl} - \omega_{il}\omega_{jk}), \tag{1.5}$$

where $N^\beta$ are the components of the unit normal vector field of $M_n$ [4].

## 2. Relationship Between the Connections $\nabla$ and $\nabla'$

It is well known that the connection coefficients of a Riemannian space whose metric tensor is $g$ are given by [5]

$$\Gamma^l_{ij} = \frac{1}{2}g^{lh}\left(\partial_i g_{jh} + \partial_j g_{ih} - \partial_h g_{ij}\right), \qquad \partial_k = \frac{\partial}{\partial x^k}. \tag{2.1}$$

Replacing $g$ in (2.1) by the metric tensor $g'$ of $M'_n$ given by (1.1), after necessary calculations, we first find the connection coefficients $\Gamma'^l_{ij}$ of $M'_n$ as

$$\Gamma'^l_{ij} = \frac{1}{2}e^{2\sigma}g'^{lk}\left(\partial_j\omega_{ik} + \partial_i\omega_{jk} - \partial_k\omega_{ij}\right) + (\partial_j\sigma)\delta^l_i + (\partial_i\sigma)\delta^l_j - (\partial_k\sigma)g'^{lk}g'_{ij}. \tag{2.2}$$

On the other hand, for the covariant derivative of the second fundamental tensor $\omega$ of $M_n$, we have [3, 4]

$$\nabla_i\omega_{jk} = \partial_i\omega_{jk} - \Gamma^h_{ij}\omega_{hk} - \Gamma^h_{ik}\omega_{jh}. \tag{2.3}$$

As a result of cyclic permutations of the indices $i$, $j$, and $k$, we obtain two more equations:

$$\nabla_j\omega_{ki} = \partial_j\omega_{ki} - \Gamma^h_{ij}\omega_{hk} - \Gamma^h_{kj}\omega_{ih}, \tag{2.4}$$

$$\nabla_k\omega_{ij} = \partial_k\omega_{ij} - \Gamma^h_{ki}\omega_{hj} - \Gamma^h_{kj}\omega_{ih}. \tag{2.5}$$

Subtracting (2.5) from the sum of (2.3) and (2.4) and using the Codazzi equations (1.4), we find

$$\nabla_i\omega_{jk} = \partial_i\omega_{jk} + \partial_j\omega_{ik} - \partial_k\omega_{ij} - 2\omega_{hk}\Gamma^h_{ij}. \tag{2.6}$$

In view of (2.6), relation (2.2) turns into

$$\Gamma'^{l}_{ij} = \Gamma^{l}_{ij} + \delta^{l}_{i}\partial_j\sigma + \delta^{l}_{j}\partial_i\sigma - g'^{lk}g'_{ij}\partial_k\sigma + \frac{1}{2}e^{2\sigma}g'^{lk}\nabla_i\omega_{jk}. \tag{2.7}$$

Relation (2.7) is the desired relation for the connection coefficients of $M_n$ and $M'_n$.

## 3. Geodesic Mappings of $M_n$ upon $M'_n$

If the map $f$ defined by (1.2) transforms every geodesic in $M_n$ into a geodesic in $M'_n$, then $f$ is called a geodesic mapping of $M_n$ into $M'_n$.

The hypersurfaces $M_n$ and $M'_n$ are in geodesic correspondence if and only if the respective connection coefficients $\Gamma^{h}_{ij}$ and $\Gamma'^{h}_{ij}$ of $M_n$ and $M'_n$ satisfy the relation [3]

$$\Gamma'^{i}_{jk} = \Gamma^{i}_{jk} + \delta^{i}_{j}\psi_k + \delta^{i}_{k}\psi_j, \tag{3.1}$$

where $\psi_k$ are the components of some 1-form which is known to be a gradient.

We first prove the following lemma which is necessary for our subsequent presentation.

**Lemma 3.1.** *Let $M_n$ and $M'_n$ be hypersurfaces of the space form $\bar{M}_{n+1}$ and let the metric tensor of $M'_n$ be defined by (1.1). If $M_n$ and $M'_n$ are in geodesic correspondence, then the 1-form $\psi_k$ is the gradient of $2\sigma$.*

***Proof.*** Since $\nabla'$ is a metric connection, we have

$$0 = \nabla'_k g'_{ij} = \partial_k g'_{ij} - g'_{lj}\Gamma'^{l}_{ik} - g'_{li}\Gamma'^{l}_{jk}.$$

Hence, with the help of (1.1) and (3.1), we obtain

$$0 = 2\omega_{ij}\partial_k\sigma + \nabla_k\omega_{ij} - 2\psi_k\omega_{ij} - \psi_i\omega_{kj} - \psi_j\omega_{ki}. \tag{3.2}$$

Changing the order of the indices $j$ and $k$ in (3.2), we find

$$0 = 2\omega_{ik}\partial_j\sigma + \nabla_j\omega_{ik} - 2\psi_j\omega_{ik} - \psi_i\omega_{kj} - \psi_k\omega_{ji}. \tag{3.3}$$

Subtracting (3.3) from (3.2) and setting

$$\phi_k = \psi_k - 2\partial_k\sigma \tag{3.4}$$

in (3.3), we conclude that

$$\omega_{ij}\phi_k - \omega_{ik}\phi_j = 0, \tag{3.5}$$

where we have used the Codazzi equations (1.4).

Note that, since $\psi_k$ is a gradient, it follows from (3.4) that $\phi_k$ is also a gradient. Multiplying (3.5) by $e^{2\sigma}$ and using (1.1), we obtain

$$\phi_k g'_{ij} - \phi_j g'_{ik} = 0. \tag{3.6}$$

At the same time, multiplying (3.6) by $g'^{ij}$ and finding the sum with respect to $i$ and $j$, we conclude, for $n > 1$, that

$$\phi_k = 0. \tag{3.7}$$

The combination of (3.4) and (3.7) yields $\psi_k = 2\partial_k \sigma$.

We now prove the following theorem:

**Theorem 3.1.** *The hypersurface $M_n$ of a space form $\bar{M}_{n+1}$ is totally umbilical if and only if $M_n$ can be geodesically mapped upon $M'_n$.*

***Proof.*** *Sufficiency.* Let $\gamma$ be a geodesic through the point $p \in M_n$ defined by $x^i = x^i(s)$ and let $s$ be the arc length of $\gamma$. Then the normal curvature, say $\kappa_n$, of $M_n$ in the direction of $\gamma$, i.e., in the direction of $\frac{dx^i}{ds}$, is given by the formula [4]

$$\kappa_n = \omega_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds}. \tag{3.8}$$

Multiplying (3.2) by $\frac{dx^i}{ds} \frac{dx^j}{ds} \frac{dx^k}{ds}$, finding the sum with respect to $i$, $j$, and $k$, , and using (3.8), we obtain

$$2\kappa_n(\partial_k \sigma)\frac{dx^k}{ds} + (\nabla_k \omega_{ij})\frac{dx^k}{ds}\frac{dx^i}{ds}\frac{dx^j}{ds} - 2\left(\psi_k \frac{dx^k}{ds}\right)\kappa_n - \left(\psi_i \frac{dx^i}{ds}\right)\kappa_n - \left(\psi_j \frac{dx^j}{ds}\right)\kappa_n = 0. \tag{3.9}$$

Since $\psi_k$ is a gradient, there exists a differentiable function $\psi$ such that $\psi_k = \partial_k \psi$. On the other hand, differentiating (3.8) covariantly in the direction of $\gamma$ and using the Frenet's formula [3]

$$\left(\nabla_k \frac{dx^i}{ds}\right)\frac{dx^k}{ds} = \kappa_g \underset{1}{\eta}^i,$$

where $\kappa_g$ is the geodesic curvature and $\underset{1}{\eta}$ is the unit principal normal vector field of $\gamma$ relative to $M_n$, we find

$$(\nabla_k \omega_{ij})\frac{dx^k}{ds}\frac{dx^i}{ds}\frac{dx^j}{ds} = \frac{d\kappa_n}{ds} - 2\kappa_g\, \omega_{ij}\, \underset{1}{\eta}^i\, \frac{dx^j}{ds}. \tag{3.10}$$

We now use relation (3.10) in (3.9) and recall that $\gamma$ is a geodesic $(\kappa_g = 0)$ in $M_n$. This yields

$$\left[\frac{\partial \kappa_n}{\partial x^i} + \left(2\frac{\partial \sigma}{\partial x^i} - 4\frac{\partial \psi}{\partial x^i}\right)\kappa_n\right]\frac{dx^i}{ds} = 0,$$

or

$$\left[\frac{\partial}{\partial x^i}\left(\ln|\kappa_n| + 2\sigma - 4\psi\right)\right]\frac{dx^i}{ds} = 0 \tag{3.11}$$

along $\gamma$.

On the other hand, by (1.1) and (3.11), we find

$$ds'^2 = {g'}_{ij}dx^i dx^j = e^{2\sigma}\omega_{ij}dx^i dx^j = e^{2\sigma}\omega_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}ds^2 = e^{2\sigma}\kappa_n ds^2,$$

whence it follows that $\kappa_n > 0$. Further, relation (3.1) implies that

$$\ln\kappa_n + 2\sigma - 4\psi = \text{const} = \mathrm{C}_1 \tag{3.12}$$

along $\gamma$.

By Lemma 3.1, $\psi = 2\sigma + C_2,\ C_2 = \text{Const}$ and, therefore, (3.12) gives

$$\kappa_n = ce^{6\sigma}, \tag{3.13}$$

where $c$ is an arbitrary positive constant.

It follows from (3.13) that the lines of curvature of $M_n$ are indeterminate at all points of $M_n$. Consequently, $M_n$ is totally umbilical.

*Necessity.* Assume that $M_n$ is a totally umbilical hypersurface of $\bar{M}_{n+1}$ which means that $\omega_{ij} = \dfrac{H}{n}g_{ij}$ where $H$ is the mean curvature of $M_n$. In this case, relation (1.1) becomes

$$g'_{ij} = \rho^2 g_{ij} \qquad \left(\rho^2 = e^{2\sigma}\frac{H}{n}\right) \tag{3.14}$$

and, hence, $M_n$ and $M'_n$ are conformal.

Relation (1.5) now implies that

$$R_{ijkl} = \left(\bar{K} + \frac{H^2}{n^2}\right)(g_{ik}g_{jl} - g_{il}g_{jk})$$

showing that $M_n$ has the constant curvature $\bar{K} + \dfrac{H^2}{n^2}$. Thus, $H$ is constant.

We now show that $M_n$ can also be geodesically mapped upon $M'_n$. Since $M_n$ is conformal to $M'_n$, their connection coefficients are related by [6]

$${\Gamma'}^h_{ij} = \Gamma^h_{ij} + \delta^h_j\rho_i + \delta^h_i\rho_j - g_{ij}\rho^h \qquad \Big(\rho_i = \nabla_i\rho,\ \ \rho^h = g^{th}\rho_t\Big). \tag{3.15}$$

To show that this conformal mapping between $M_n$ and $M'_n$ is also a geodesic mapping, according to (3.15) and (3.1) it is necessary to find a 1-form $\psi_k$ such that

$$\Gamma^h_{ij} + \delta^h_j\psi_i + \delta^h_i\psi_j = \Gamma^h_{ij} + \delta^h_j\rho_i + \delta^h_i\rho_j - g_{ij}\rho^h$$

or

$$\delta^h_j(\psi_i - \rho_i) + \delta^h_i(\psi_j - \rho_j) + g_{ij}\rho^h = 0. \tag{3.16}$$

Transvecting (3.16) by $g^{ij}$, we get

$$g^{ih}(\psi_i - \rho_i) + g^{jh}(\psi_j - \rho_j) + n\rho^h = 0$$

or

$$2g^{ih}(\psi_i - \rho_i) + n\rho^h = 0. \tag{3.17}$$

Multiplying (3.17) by $g_{hj}$ and finding the sum over $h$, we get

$$2\psi_j + (n-2)\rho_j = 0.$$

Thus, by virtue of (3.14), we find

$$\psi_j = \left(\frac{2-n}{2\sqrt{n}}\sqrt{H}\right)\partial_j e^{\sigma}, \quad H > 0.$$

With this choice of $\psi_j$, the conformal mapping mentioned above also becomes a geodesic mapping.

Theorem 1.1 is proved.

In the special case where $\sigma = 0$ throughout $M_n$, i.e., $g' = \omega$, we can mention some properties of $M_n$ which is in the geodesic correspondence with $M'_n$ :

1. From Lemma 3.1 and relation (3.1), we conclude that any geodesic mapping of $M_n$ upon $M'_n$ is connection preserving.
2. It follows from (3.13) that $M_n$ has constant normal curvature along each geodesic through a point $p \in M_n$.
3. The underlying geodesic mapping is a homothety.

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