PARALLEL IMPLEMENTATION OF ITERATIVE RATIONAL KRYLOV METHODS FOR MODEL ORDER REDUCTION

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ABSTRACT

Model order reduction (MOR) techniques are getting more important in large scale computational tasks like large scale electronic circuit simulations. In this paper we present some experimental work on multiprocessor systems for rational Krylov methods. These methods require huge memory and computational power especially in large scale simulations. Therefore, these methods are fairly suitable for parallel computing.

1. INTRODUCTION

Models of the interconnect structure of the very large scale integrated (VLSI) circuits can be given in general as a linear state space system:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t) \tag{1}$$

where $A \in \mathcal{R}^{nxn}$, $B \in \mathcal{R}^{nxm}$, $C \in \mathcal{R}^{mxn}$ and $D \in \mathcal{R}^{mxm}$ and n is the order of the system.

Model order reduction (MOR) methods are extensively studied [1]. In this work we deal with the rational Krylov based methods. If these type of methods are implemented iteratively the optimal reduced system can be obtained with respect to $\mathcal{H}_2$ norm of the system. Transfer function of a linear system can be defined as,

$$G(s) = C^*(sI - A)^{-1}B + D \tag{2}$$

where initial conditions of (1) are taken to be equal zero.

If the system triple is given as $\Sigma = (A, B, C)$, one can produce projection matrices $V \in \mathcal{R}^{nxk}$ and $W \in \mathcal{R}^{kxn}$ to obtain a kth order reduced system $\hat{\Sigma} = (\hat{A}, \hat{B}, \hat{C})$. These projection matrices have to satisfy the condition $W * V = I_k$ where I_k is the k^{th} order identity matrix. The reduced order system matrices are given below.

$$\hat{A} = W^*AV, \quad \hat{B} = W^*B, \quad \hat{C} = CV \tag{3}$$

The basic idea of MOR is to build a reduced order system with and order $k \ll n$ while the reduced system keeping some of the basic properties of the original system such as stability and passivity. Constructing these projection matrices is widely studied in literature [2]. The methods can be divided into three main types. The first family of the methods is using Krylov based moment matching techniques [3]. The goal of these methods is to match the moments of the transfer function $G(s)$ at some selected interpolation points. These type of methods are computationally efficient but have some drawbacks. They do not provide exact *a priori* knowledge on approximation error. Second type of methods are associated with the Hankel singular values of the system and they are called SVD-based methods. Although these methods have satisfactory features they require huge computational power [4]. Therefore, some hybrid methods are developed in literature. The hybrid methods, take the theoretical advantages of the SVD methods and combine them with the computationally feasible nature of the Krylov methods [5].

In this work, we consider the rational Krylov based methods. Basically rational Krylov based methods match the transfer function at differently selected interpolation points. These type of methods are widely used in current model reduction research such as passivity-preserving model reduction methods and $\mathcal{H}_2$ optimal model reduction [6]. Although the computational cost of rational Krylov methods is also high, they are suitable for parallel programming. In this work, we employ a parallel approach to produce $\mathcal{H}_2$ optimal reduced model using rational Krylov methods.

The rest of the paper is organized as follows. In second section, the methods and problem are introduced. In third section numerical results are given and in the last section, some conclusions are given.

2. DEFINITION OF THE PROBLEM

Assume that k distinct points in complex plane are selected for interpolation. Then interpolation matrices, V and W can

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be built as shown below.

$$\begin{aligned} \hat{V} &= [(s_1 I - A)^{-1} B \dots (s_k I - A)^{-1} B] \\ \hat{W} &= [(s_1 I - A)^{-1} C^* \dots (s_k I - A)^{-1} C^*] \end{aligned} \quad (4)$$

Assuming that $det(\hat{W}^* V) \neq 0$, the projected reduced system can be built as, $\hat{A} = W^T AV$, $\hat{B} = W^T B$, $\hat{C} = CV$, $\hat{D} = D$ where $V = \hat{V}$ and $W = \hat{W}(\hat{V}^* W)^{-1}$ to ensure $W^* V = I_k$. The basic problem is to find a strategy to select the interpolation points. As the worst case, the interpolation points can be selected as randomly from the operating frequency of the system. The algorithm for this case is given in Alg.1.

Algorithm 1 RANDOMLY SELECTED POINTS

Require: System matrices A, B, C and D,
Ensure: Reduced system matrices $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{D}$,

Select k distinct point from operating frequency,
Form V and W according to (4),
Compute $W = \hat{W}(\hat{V}^* W)^{-1}$ in order to ensure $\hat{W}^* V = I_k$,
$\hat{A} = W^* AV$, $\hat{B} = W^* B$, $\hat{C} = CV$, $\hat{D} = D$.

To improve this approach several methods can be used. In this work we use the iterative rational Krylov approach to achieve $\mathcal{H}_2$ norm optimal reduced model [7]. $\mathcal{H}_2$ norm of a system is defined as below.

$$||G||_2 := \left[\int_{-\infty}^{+\infty} |G(j\omega)|^2 d\omega \right]^{1/2} \quad (5)$$

Reduced order system $G_r(s)$ is $\mathcal{H}_2$ optimal if it minimizes the

$$G_r(s) = argmin_{deg(\hat{G})=r} ||G(s) - \hat{G}(s)||_{\mathcal{H}_2} \quad (6)$$

Interpolation based first order condition for the optimal $\mathcal{H}_2$ approximation is given in below theorem [8].

Theorem 1: If $G_r(s)$ solves the optimal $\mathcal{H}_2$ problem and $\hat{\lambda}_i$ values are the eigenvalues of $\hat{A}_r$ which have one multiplicity, then

$$\frac{d^k}{ds^k} G(s)|_{s=-\hat{\lambda}_i} = \frac{d^k}{ds^k} G_r(s)|_{s=-\hat{\lambda}_i}, \quad k = 0, 1. \quad (7)$$

The result given by Grimme about rational Krylov method is also important for the implementation of $\mathcal{H}_2$ optimal model reduction [9].

Theorem 2: If the system Σ and interpolation points s_i are given and V and W are obtained from (4) then the reduced system $\hat{\Sigma}$ interpolates Σ and its first derivative at s_i points.

Gugercin et.al. combine these two important results to achieve an iterative rational Krylov methods to obtain $\mathcal{H}_2$ optimal reduced order model [10]. The algorithm is given in Alg.2.

Algorithm 2 ITERATIVE RATIONAL KRYLOV

Require: System matrices A, B, C and D.
Ensure: Reduced system matrices $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{D}$.

Make an initial selection for s_i
Form V and W according to (4).
Compute $W = \hat{W}(\hat{V}^* W)^{-1}$ in order to ensure $\hat{W}^* V = I_k$.
while NOT CONVERGED **do**
    $\hat{A} = W^T AV$
    $s_i \leftarrow -\lambda_i(\hat{A})$ for $i = 1 : k$
    Form V and W according to (4).
    Compute $W = \hat{W}(\hat{V}^* W)^{-1}$ in order to ensure $\hat{W}^* V = I_k$.
end while
$\hat{A} = W^* AV$, $\hat{B} = W^* B$, $\hat{C} = CV$, $\hat{D} = D$

When the eigenvalues of the A_k matrices converge the algorithm will stop. We use a ladder RLC network as benchmark example for the numerical implementation of the Alg.1 and Alg.2. Minimal realization of the circuit is given in Fig.1. For this circuit order of the system $n = 5$. On the other hand, system matrices of this circuit can easily be extended [11]. In the frequency plots given in Fig.2, order of the system n is taken as 201 and the reduced system order k is taken as 40 for both methods.

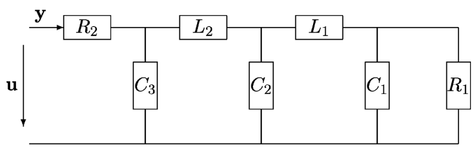

Fig. 1: 5^{th} order minimal realization of the RLC circuit used in experiments.

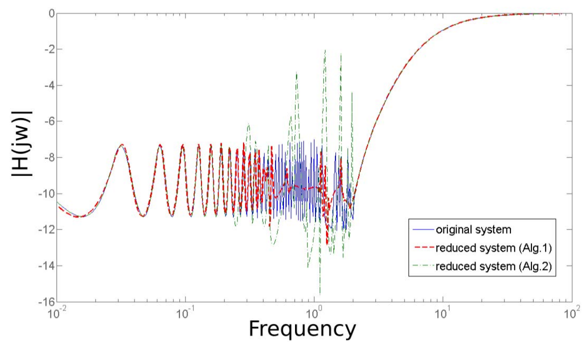

Fig. 2: Frequency plots of the reduced and original systems with both methods.

3. PARALLEL IMPLEMENTATION OF THE METHOD

Computational cost of the rational Krylov methods can be given as $\mathcal{O}(kn^3)$ for dense problems where k is the number of interpolation points. In Alg.2 rational Krylov methods are used iteratively and the computational complexity has to be multiplied by the iteration number r. Although both algorithms have k times factorization to compute $(s_i I - A)^{-1}B$, these factorizations can be computed on different processors independently. Also the matrix-matrix and matrix-vector multiplications in the algorithms are amenable to parallel processing. Alg. 1. is the basic computationally expensive part of the Alg. 2. Therefore, we parallelize Alg.1. first and then apply parallelized rational Krylov part in Alg.2. Parallel version of the Alg.1. is given in Alg.3.

Algorithm 3 PARALLEL VERSION OF ALG.1

Require: System matrices A, B, C and D,
Ensure: Reduced system matrices $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{D}$,

1: Select k distinct point s_i from operating frequency,
2: Distribute s_i to processors,
3: Form V and W according to (4) in each processor,
4: Compute $W = \hat{W}(\hat{V}^*W)^{-1}$ in order to ensure $\hat{W}^*V = I_k$ in parallel,
5: $\hat{A} = W^*AV$, $\hat{B} = W^*B$, $\hat{C} = CV$, $\hat{D} = D$.

This parallel version of algorithm can be easily used in Alg.2 with a small change. All codes are written in Matlab language and for the parallelization we use Matlab's parallel computing toolbox [12]. In Table 1 and 2, n is the order of the original system.

Table 1: CPU times of parallel version of Alg.1 for different system orders.

Proc no.	time (n=2000)	time (n=5000)
1	59.8	1485.3
2	31.4	780.7
4	21.2	451.4
8	23.8	374.2

Table 2: CPU times of parallel version of Alg.2 for different system orders.

Proc no.	time (n=2000)	time (n=5000)
1	512.6	2486.2
2	410.7	1605.9
4	203.9	810.8
8	176.1	648.4

Speedup graphs of the algorithms are given in Fig.3. It can easily be seen from the figures, when we increase the number of processors processing time decreases appreciably up to some point, after which it starts to increase. This is due to communication times becoming dominant over computation time. But in both algorithm, when the size of the system matrices better speedups are obtained.

Speedup of a parallel algorithm is defined by the formula below.

$$S_p = \frac{T_1}{T_p} \tag{8}$$

Speedup graphics of the algorithms are given in Fig.3 and Fig.4 respectively.

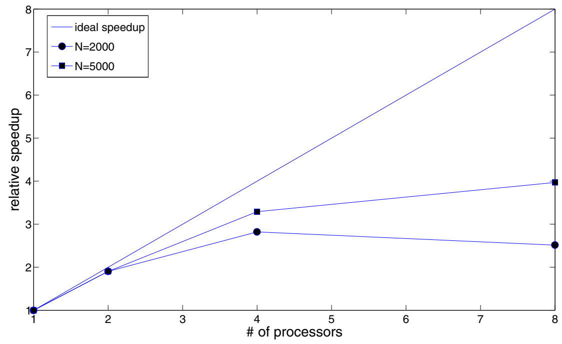

Fig. 3: Speedup graphics of the Alg.1 for different system order n.

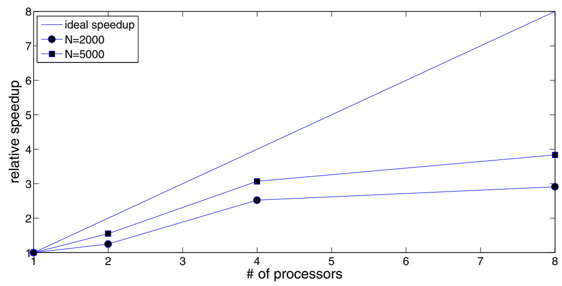

Fig. 4: Speedup graphics of the Alg.2 for different system order n.

4. CONCLUSION

In this work, iterative rational Krylov method based optimal $\mathcal{H}_2$ norm model reduction methods are parallelized. These methods need huge computational demands but the algorithm itself is suitable for parallel processing. Therefore, computational time decreases when the number of processors is increased. Due to communication needs of the processors, communication time dominates the overall process time when the system order is small. But in larger orders, parallel algorithm has better speedup values.

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