Partitioning 3-arcs into Steiner Triple Systems

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Abstract: In this article, it is shown that there is a partitioning of the set of 3-arcs in a projective plane of order three into nine pairwise disjoint Steiner triple systems of order 13. © 2017 Wiley Periodicals, Inc. J. Combin. Designs 25: 581–584, 2017

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1. INTRODUCTION AND THE MAIN RESULT

Let X be a set of n $(n \geq 3)$ points and B a collection of 3-subsets of X such that every 2-subset of X is covered in at most one member of B, then the system (X, B) is called a *packing triple system* of order n. Such a system is said to be *optimal* if there is no any other packing triple system of order n with a larger collection of 3-subsets. A Steiner triple system $STS(X, B)$ is an optimal packing triple system of order n in which every 2-subset of X is contained in exactly one member of B. A Steiner triple system of order n exists if and only if $n \equiv 1, 3$ $(mod\ 6)$ [5]. A set of $n - 2$ pairwise disjoint Steiner triple systems of order n is called a *large set* in which the maximum number of such systems is attained. Existence (or nonexistence) of large sets has been long studied and, in particular, Lu showed [2, 3] the existence of large sets of Steiner triple systems for all $n \equiv 1$ or 3 (mod 6), $n \neq 7$. However, Lu's work was missing the cases for $n \in \{141, 283, 501, 789, 1, 501, 2, 365\}$ and Teirlinck [4] completed these cases.

Donald L. Kreher asked the following interesting problem in 2011 during one of his talks on transverse t-designs: Can the noncollinear triples of a projective plane of order 3 be partitioned into disjoint Steiner triple systems? There are 234 noncollinear triples, i.e. 3-arcs, in a plane of order 3 and a Steiner triple system of order 13 contains exactly 26 triples. Therefore, such a partitioning would contain exactly nine Steiner triple systems of order 13 and the answer is in the affirmative as presented in Theorem 1.1.

TABLE I. Triples in C_1

1	2	5	8	9	12	1	3	8	2	8	10
2	3	6	9	10	13	2	4	9	3	9	11
3	4	7	1	10	11	3	5	10	4	10	12
4	5	8	2	11	12	4	6	11	5	11	13
5	6	9	3	12	13	5	7	12	6	12	1
6	7	10	1	4	13	6	8	13	7	13	2
7	8	11				1	7	9			

Theorem 1.1. *There is a partitioning of the set of 3-arcs in a projective plane of order 3 into nine pairwise disjoint cyclic (or noncyclic) Steiner triple systems of order 13.*

Moreover, there are 52 collinear triples and the set of these collinear triples cannot be partitioned into two disjoint Steiner triple systems of order 13, since a line of the plane generates four triples of which any two cannot be in a Steiner triple system at the same time.

2. THE PROOF OF THEOREM 1.1

We assume the reader is familiar with basic definitions related to group actions and combinatorial designs including finite projective planes.

Let X be a nonempty set of size n, then we denote by $G|X$ the action of the group G on X. For $x \in X$ and $g \in G$, x^g denotes the image of x under g. If $S \subset X$, then we define that $S^g = \{x^g \mid x \in X\}$. Moreover, if $U \subset 2^X$, then we let $U^g = \{S^g \mid S \in U\}$. An *automorphism* of a combinatorial design (X, B) is a bijection $\alpha : X \to X$ such that $S^\alpha \in B$, $S \in B$. In particular, let S_X be the symmetric group defined on set X, then an $STS(X, B)$ is said to be *cyclic* if there is an automorphism $\alpha \in S_X$ with a single cycle of length n.

It is well known that there are up to isomorphism exactly two non-isomorphic Steiner triple systems of order 13, one of which is cyclic [1]. Let $X = \{1, 2, \ldots, 13\}$, and $g_0 = (1, 2, 3, \ldots, 13)$, then developing the base blocks $\{1, 2, 5\}$ and $\{1, 3, 8\}$ with the cyclic group generated by g_0 gives rise to the cyclic system C_1 in Table I.

Let $g_1 := (1, 4, 8)(3, 10, 12)(5, 7, 11)$ and $g_2 := (1, 4, 8)(2, 6, 13)(3, 12, 10)$ be permutations in S_X, then define the group $H_1 := \langle g_1, g_2 \rangle$ of order 9. Further, let us define $\mathscr{P}_1 = \{{C_1}^h \mid h \in H_1\}$, then (X, P) defines a cyclic Steiner triple system of order 13 for any $P \in \mathscr{P}_1$. Moreover, $\mathscr{P}_1$ gives rise to exactly nine pairwise disjoint triple systems, since $P \cap R = \emptyset$ whenever $P \neq R$, where $P, R \in \mathscr{P}_1$. There are 234 triples contained in $\mathscr{P}_1$ and exactly 286 triples in 2^X. The remaining 52 triples come from the lines of π the projective plane of order 3, if $\mathscr{P}_1$ results in an organization of the 3-arcs of π into pairwise disjoint Steiner triple systems of order 13. Therefore, the next step is to partition the set of these into 13 parts of size 4 such that the union of the four triples in each part is a 4-subset. Such a partitioning is given in Table II. This establishes Lemma 2.1.

Lemma 2.1. *The set of 3-arcs in a projective plane of order 3 can be partitioned into nine pairwise disjoint cyclic Steiner triple systems of order 13.*

TABLE II. 4-subsets defining π

1	2	3	7	1	4	8	9	1	5	6	10
1	11	12	13	2	4	10	11	2	5	8	12
2	6	9	13	3	4	5	13	3	6	8	11
3	9	10	12	4	6	7	12	5	7	9	11
7	8	10	13								

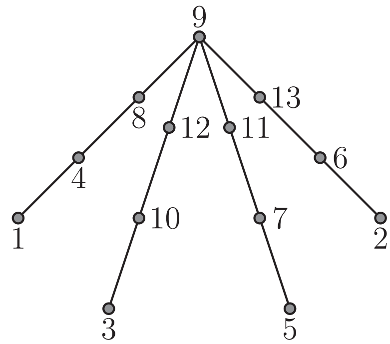

FIGURE 1. Lines of π through the point 9

Note that H_1 is a collineation group for π, since both the generators g_1 and g_2 preserve its structure. In particular, each of g_1 and g_2 fixes the points of the lines (pointwise or setwise) through the point 9. See Figure 1.

Now let us consider the noncyclic case. A computer search finds a noncyclic Steiner triple system of order 13 from the 3-arcs of π. The set C_2 of these triples is given below in Table III.

Similar to the cyclic case, let us define $g_3 := (1, 4, 9)(2, 12, 5)(3, 6, 11)$ and $g_4 := (1, 4, 9)(3, 11, 6)(7, 10, 13)$ whose cycles are determined from the lines of π through the point 8 (See Fig. 2). Moreover, if $H_2 := \langle g_3, g_4 \rangle$, then H_2 is of order 9 and preserves the structure of π. Hence, it is a collineation group for π. Let us also define $\mathscr{P}_2 = \{{C_2}^h \mid h \in H_2\}$, then $\mathscr{P}_2$ is a partitioning of 3-arcs into nine disjoint Steiner triple systems of order 13. This establishes Lemma 2.2.

TABLE III. Triples in C_2

8	11	12	3	8	13	4	7	9	4	8	10
3	6	12	1	6	8	2	7	12	9	12	13
4	5	12	1	3	9	1	5	11	2	3	5
1	2	4	5	10	13	6	7	11	2	11	13
5	6	9	1	10	12	4	6	13	2	8	9
9	10	11	5	7	8	1	7	13	3	7	10
2	6	10	3	4	11						

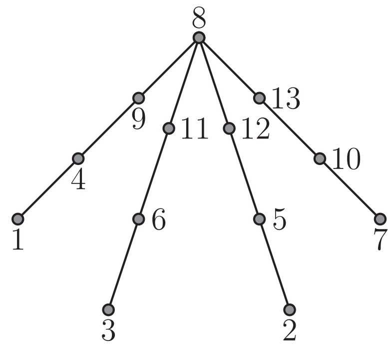

FIGURE 2. Lines of π through the point 8

Lemma 2.2. *The set of 3-arcs in a projective plane of order 3 can be partitioned into nine pairwise disjoint noncyclic Steiner triple systems of order 13.*

As discussed above, there are two Steiner triple systems of order 13, up to isomorphism, one of which is cyclic and other noncyclic. In a projective plane of order 3, there are 234 many 3-arcs that can be partitioned into pairwise disjoint Steiner triple systems (as given in Lemmas 2.1 and 2.2), so Theorem 1.1 follows.

REFERENCES

[1] C. J. Colbourn and J. H. Dinitz, Handbook of Combinatorial Designs, Chapman and Hall/CRC, Boca Raton, 2006.

[2] J. X. Lu, On large sets of disjoint Steiner triple systems I, II, III, J Comb Theory Series A 34 (1983), 140–146, 147–155, 156–183.

[3] J. X. Lu, On large sets of disjoint Steiner triple systems IV, V, VI, J Comb Theory Series A 37 (1984), 136–163, 164–188, 189–192.

[4] L. Teirlinck, A completion of Lu's determination of the spectrum for large sets of disjoint Steiner systems, J Comb Theory Series A 57, (1991), 302–305.

[5] T. P. Kirkman, On a problem in combinations, Cambridge Dublin Math J 2 (1847), 191–204.

[6] L. Heffter, Über Nachbarconfigurationen, Tripelsysteme und metacyklische Gruppen, Deutsche Mathem Vereining Jahresber 5 (1896), 67–69.

[7] R. Peltesohn, Eine Lösung der beiden Heffterschen Differenzenprobleme, Compositio Math 6 (1939), 251–257.